MATHEMATICAL MODELING EXAM
10–12–2015

- *Mark all papers with name and student number.*
- *The maximum time for the exam is four hours.*
- *Use of lecture notes is NOT allowed.*

QUESTION 1

Consider a parasitoid-host system where each adult butterfly (i.e., the host) produces new larvae at a constant rate. If a larva is found by a parasitoid wasp, then the wasp deposits a single egg inside the larva, independently of the number of eggs that may already be inside. A larva containing zero eggs of the parasitoid develops into an adult butterfly, but a larva containing a single egg develops into an adult parasitoid. Larvae with two or more eggs inside do not develop into anything at all but simply die (presumably because there is not enough food present for the parasitoids to complete their development). Larvae, butterflies and parasitoids are also subject to death at a constant rate due to other (unspecified) causes.

Formulate a model of the above system, i.e., specify the various i-states, model the i-level processes with a network of mono- and bimolecular reactions, and assuming mass-action, give the corresponding population model as a system of differential equations for the population densities.

QUESTION 2

Consider the following SIS-model:

$$\begin{cases} \frac{dS}{dt} &= \alpha - \beta SI + \gamma I - \delta S \quad \text{(susceptibles)} \\ \frac{dI}{dt} &= \beta SI - \gamma I - (\delta + \varepsilon) I \quad \text{(infected)} \end{cases}$$

with all positive parameters. Interpret the various terms and parameters of the above system in terms of i-level processes. Give a phase-plane analysis of the system. Find all equilibria and establish their stability. Use linear stability analysis if the phase-plane analysis is inconclusive. Distinguish between the cases where there *does* and *does not* exists a positive equilibrium with both S and I present.

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QUESTION 3

The following discrete-time model is due to Hassell (1975) and is often being used for insect populations:

$$x_{n+1} = \frac{ax_n}{(bx_n + 1)^c}$$

where $a, b, c > 0$ and where x_n denotes the population density at the beginning of the season in year n. Show that this model can be derived from a continuous-time interference competition model for the within-season dynamics, assuming there is a single reproductive burst at the beginning of the season, and assuming that during the season adult individuals attack both other adults and juveniles (i.e., eggs or larvae). No other processes are involved. Give the within-season model as a system of two ordinary differential equations with initial conditions. Solve these equations and derive the between-season dynamics. Show that the resulting model is Hassell's model. In particular, express the a, b, c in terms of the the parameters of the within-season model.

QUESTION 4

Formulate a model of the following situation as a system of partial differential equation with reflecting boundary conditions:

When two individuals meet they may start a fight during which they ran around as a pair in a random manner. The fight lasts for an exponentially distributed amount of time until they separate again. In order to avoid painful collisions, single individuals tend to move away from fighting pairs. Sometimes, however, a collision is unavoidable, in which case the fight is instantly over and the pair breaks immediately apart.

Don't forget the boundary conditions.

Success!