Inverse Problems 1 2022 - Final Exam

October 2022

Instructions

1. This exam is worth a total of 40 points.
2. You have 3 hours to answer it.
3. Answer the questions in a separate sheet of paper. **Write your name and student number to the each sheet of paper.** Annotations made in this sheet will not be considered.
4. Question 1 is mandatory. You can choose 3 of the remaining 4 questions (Q2 to Q5) to answer. **Make it very clear in your answer sheet which questions should be considered**. Anything you write about the excluded question will not be considered.

Q1 (Mandatory) (Total points: 10) Consider the Hadamard's classical definition of a well-posed problem.

Well-posedness (Hadamard). *A problem is well-posed if all the three conditions below holds*

(H1) Given any input, there exists an output;

(H2) For each input, the output is unique;

(H3) The output depends continuously on the input.

In this course we considered the measurement model

$$m = Af,$$

where $f, m \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$ is a convolution matrix. Answer the following questions:

(a) (2 points) What is the "input" and "output" in the direct problem (convolution)?

(b) (2 points) What is the "input" and "output" in the inverse problem (deconvolution)?

(c) (2 points) Explain why the direct problem is well-posed in the sense of (H1)—(H3).

(d) (2 points) Consider the singular value decomposition $A = UDV^T$. Write down the important properties of the matrices U, V and D.

(e) (2 points) Assume that the singular values of A are all positive. Explain how the inverse problem (deconvolution) can be ill-posed. (HINT: Reformulate (H3) like this: if the input changes a little bit, the output also changes only a little bit.)

Q2 (Total points: 10) Consider the following discrete point spread function $p[n]$ of length 5 and original signal $s[n]$ of length 10

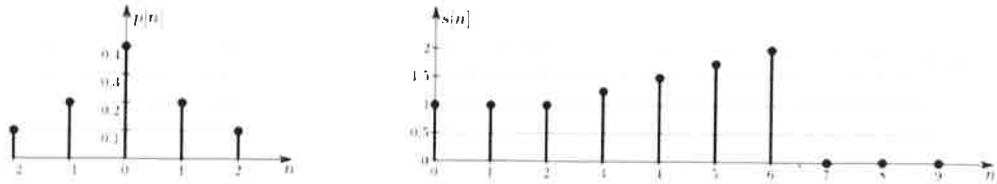

Figure 1: Point spread function $p[n]$ and signal $s[n]$.

Create the convolution matrix A such that $p * s = As$ for each of the following boundary conditions. Make sure the matrix is square.

a) (3 points) Zero boundary condition

$$s[n] = \begin{cases} 0 & n < 0 \\ 0 & n > 9 \end{cases}$$

b) (3 points) Periodic boundary condition

c) (4 points) Constant boundary condition

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$$s[n] = \begin{cases} s[0] & n < 0 \\ s[9] & n > 9 \end{cases}$$

0

p_{-2} p_{-1} p_0 p_1 p_2

s_1 s_1 s_1 s_2 s_3 . . .

0.7

$(p_{-2}+p_{-1}+p_0)s_1$

0.3

$(p_{-1}+p_0)s_1$

$s_{10}(p_1+p_2)$

$s_{10}(p_0+p_1+p_2)$

s_{10} s_{10} s_{10}

p_{-2} p_{-1} p_0 p_1 p_2

Q3 (Total points: 10) Consider the linear system $Ax = b$, where $A \in \mathbb{R}^{n \times n}$ is invertible and $b \neq 0 \in \mathbb{R}^n$, leading to the solution $x = A^{-1}b \in \mathbb{R}^n$. Also consider the noisy measurement $b_\delta = b + \delta$, leading to the perturbed solution $x_\delta = A^{-1}b_\delta \in \mathbb{R}^n$.

We will proof this important relation between the relative error in the solution $||x - x_\delta||/||x||$ and the relative noise in the measurement vector $||\delta||/||b||$

$$\frac{||x - x_\delta||}{||x||} \leq \text{cond}(A)\frac{||\delta||}{||b||}, \tag{1}$$

where $||\cdot||$ is assumed to be the Euclidean norm and $||b|| > 0$ and $||x|| > 0$.

HINTS FOR THIS QUESTION:

- $||Mv|| \leq ||M||||v||$, where M and v are a matrix and a vector of appropriate sizes.
- The euclidean norm of a matrix (spectral norm) is the largest singular value of the matrix

$$||A|| = \sigma_{max}.$$

a) (2.5 points) Show that

$$||x|| \geq \frac{||b||}{\sigma_1}, \tag{2}$$

where σ_1 is the largest singular value of A.

b) (2.5 points) Next we need to find $||A^{-1}||$. Show that

$$||A^{-1}|| = \frac{1}{\sigma_n}, \tag{3}$$

where σ_n is the smallest singular value of A.

c) (2.5 points) Show that

$$||x - x_\delta|| \leq \frac{1}{\sigma_n}||\delta||. \tag{4}$$

d) (2.5 points) Finally, show that

$$\frac{||x - x_\delta||}{||x||} \leq \text{cond}(A)\frac{||\delta||}{||b||}, \tag{5}$$

Q4 (Total points: 10) Consider the following linear system $Ax = b$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1+\epsilon \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \tag{6}$$

where $\epsilon > 0$ is a positive real number. And the following theorem

Theorem. *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix. The singular values of A are the absolute values of the eigenvalues of A and the columns of $U = V$ are the eigenvectors of A.*

a) (2.5 points) Show that the singular values of A are

$$\sigma_1 = 1 + \frac{\epsilon}{2} + \sqrt{1 + \left(\frac{\epsilon}{2}\right)^2} \tag{7}$$

$$\sigma_2 = 1 + \frac{\epsilon}{2} - \sqrt{1 + \left(\frac{\epsilon}{2}\right)^2}. \tag{8}$$

b) (2.5 points) Find the condition number (with respect to the Euclidean norm) of matrix A when $\epsilon = 0.1$.

c) (2.5 points) Assume you have noisy measurements $b_\delta = b + \delta$. Using (1) from question 3, find the maximum norm of the measurement error $\|\delta_{max}\|$ that still guarantees that the relative error $\|x - x_\delta\|/\|x\|$ is smaller or equal to 0.01. Assume $\epsilon = 0.1$.

d) (2.5 points) If $\epsilon < 0.1$, do you expect $\|\delta_{max}\|$ to be bigger or smaller than in the previous item? Justify your answer.

Q5 (Total points: 10) Consider the same linear system of the previous question with noisy measurement $Ax = b_\delta$, where $\epsilon = 0.1$ and

$$b_\delta = b + \delta = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} = \begin{bmatrix} 1.1 \\ 2.1 \end{bmatrix}, \tag{9}$$

HINTS FOR THIS QUESTION:
The inverse of a 2x2 matrix:

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \rightarrow \quad M^{-1} = \frac{1}{\det(M)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{10}$$

a) (5 points) Solve the linear system adopting the following Tikhonov regularization strategy and assuming $\alpha = 0.1$. Use the normal equations formulation

$$x_\delta = \arg\min_x \{\|Ax - b_\delta\|^2 + \alpha\|x\|^2\}. \tag{11}$$

$T_\alpha = (A^T A + \alpha I)^{-1}(A^T b_\delta)$

b) (5 points) Compute the residual of the solution you found in the previous item. Following the Morozov's discrepancy principle, is $\alpha = 0.1$ above or below the optimal value? Justify your answer. OBS: If you did not find any result in the previous item, assume $x_\delta = [0.3 \quad 0.8]^T$ for your calculations.