Introduction to Mathematical Biology
EXAM: 31 October 2022

1. The exam lasts 105 minutes (12.15-14.00).
2. On the first page, please write your name and email readably (e.g. in print).
3. Put your name on every separate paper.
4. You can ask me if you do not understand a question only because of the English.
5. I shall email you the results, and if you wish, you can have a look at the corrected exam. If you do not hear about the results within two weeks, please contact me.
6. You can re-take the exam if you want a better grade.

The three problems are given equal weight in the grade.

1. Many proteins regulate their own production by binding to their gene and blocking the transcription of DNA (negative feedback). Let x and y denote respectively the concentration of the protein and the fraction of time the gene is free (this is similar to the concentration of free genes). When the gene is free, it produces the protein at a constant speed c. The protein binds to the gene at rate k_1 and releases from the gene at rate k_2. The unbound protein decays at rate δ. These assumptions lead to the ODE system

protein   free   bound

$$\frac{dx}{dt} = cy - k_1xy + k_2(1-y) - \delta x = y(c - k_1 x - k_2) + k_2 - \delta x$$

free

$$\frac{dy}{dt} = -k_1xy + k_2(1-y)$$

where all parameters are assumed to be positive. Assume that binding and unbinding is much faster than the production and decay of the protein, i.e., $k_1, k_2 \gg c, \delta$.

(a) Find the quasi-equilibrium of y and show that it is always positive and globally asymptotically stable.
(b) Write a single differential equation for the slow dynamics of protein concentration. Investigate the number and stability of equilibria.

2. Consider a population of an annual plant A that follows the Skellam model of discrete-time population growth,

$$x_{t+1} = 1 - e^{-\alpha x_t}$$

where x_t is the fraction of occupied sites in year t and α is the number of seeds per plant. This model describes a population where seeds are distributed randomly over the sites and each site that has received at least one seed will be occupied by one plant.

Suppose that this populaton is at a positive equilibrium. Then a new plant species B appears with the following properties:

(i) it has β seeds per plant, and
(ii) if there is any seed of the original species A present in a site, then all seeds of species B in this site die (i.e., they are all outcompeted by A)

(a) Show that plant A is viable if $\alpha > 1$ and in this case, its positive equilibrium $\hat{x}$ is stable. *Hint:* the value of $\hat{x}$ cannot be obtained analytically but this does not prevent the proof. Part (b) can be done without (a).
(b) Show that if β is sufficiently large, then the two species mutually invade each other's equilibrium population and therefore coexist.

3. Consider a population where the reproduction of females is limited by their ability to find males. The population has an equal number of males and females (N each). A female encounters males according to mass action at a rate β. Upon mating, the female produces B female and B male offspring and returns to searching for males after T time. All individuals die at a density-dependent rate $\mu + aN$. By the analogy of the Holling II functional response for the birth term, these assumptions lead to the dynamics

$$\frac{dN}{dt} = \frac{bN}{1+cN}N - (\mu + aN)N$$

where $b = \beta B$, $c = \beta T$, and all parameters are assumed to be positive.

(a) Find all biologically meaningful equilibria of this model and establish their stability. *Hint:* recall that finding the stability of one equilibrium helps to find the stability of all others.
(b) Suppose fecundity and therefore b is initially high enough to maintain a positive population size but to a change in the environment, b decreases. Describe both mathematically (bifurcation) and in practical terms (verbal advice to a manager) what happens to this population and what is the minimum value of b to avoid extinction. (Non-degeneracy conditions need not be checked.)