

Introduction to Mathematical Biology

EXAM: 24 October 2022

1. The exam lasts 105 minutes (10.15-12.00).
2. On the first page, please write your name and email readably (e.g. in print).
3. Put your name on every separate paper.
4. You can ask me if you do not understand a question only because of the English.
5. I shall email you the results, and if you wish, you can have a look at the corrected exam. If you do not hear about the results within two weeks, please contact me.
6. You can re-take the exam if you want a better grade.

The three problems are given equal weight in the grade.

1. Suppose that individuals encounter each other according to mass action at a rate β . An encounter means a fight, in which one of the two individuals dies with probability p_1 and both die with probability p_2 ($1 - p_1 - p_2$ is the probability that both survive). The natural death rate (all deaths other than fights) is μ and the birth rate is b (both are constant, $0 < \mu < b$).

(a) Calculate the size of the population at the nontrivial equilibrium and show that this equilibrium is stable.

(b) Calculate the probability that an individual dies in a fight when the population is at equilibrium.

2. Consider a population of bacteria that pollute the environment by producing a toxic substance. The dynamics is governed by

$$\begin{aligned}\frac{dN}{dt} &= bN - (\mu + \rho T)N \\ \frac{dT}{dt} &= \alpha N - \delta T,\end{aligned}$$

where $N(t)$ is the size of the population at time t . $T(t)$ is the concentration of the toxin, which is produced by the bacteria at a *per capita* rate $\alpha > 0$ and decays exponentially at a rate $\delta > 0$ (uptake by the bacteria is negligible). The toxin kills at a rate proportional to its concentration, $\rho T(t)$. Finally, $\mu > 0$ and $b > \mu$ are the (constant) background death rate and the birth rate, respectively. We assume that the toxin is produced and decays fast compared to the population dynamics of the bacteria (α and δ are large compared to

b , μ and ρ).

(a) Show that the bacteria grow according to the logistic equation and give r_0 and K in terms of the model parameters.

(b) Suppose several strains of the bacteria are grown together, which all produce the same toxin but differ in the parameters b and μ . Show that the model is an optimization model and the optimal strain maximizes K . (Results derived in the lecture can be used without proof.)

3. Insect pests may be eradicated by releasing sterilized males. Females who mate with a sterile male do not produce any offspring, so that the average *per capita* fecundity of females decreases. Assume that in absence of control, the population follows the Beverton-Holt model

$$N_{t+1} = \frac{bN_t}{1 + aN_t}$$

with $b > 1$ and $a > 0$, where N_t denotes the number of females. Let S be the number of sterile males released. We can reasonably assume that without interference, the number of males and females are equal, so that the number of ordinary fertile males is also N_t . Females fail to reproduce if they mate with a sterile male, and this happens with probability $S/(N_t + S)$. The population dynamics thus become

$$N_{t+1} = \frac{bN_t}{1 + aN_t} \left(1 - \frac{S}{N_t + S}\right) = \frac{bN_t}{1 + aN_t} \cdot \frac{N_t}{N_t + S}$$

It is straightforward to show that the model is undercompensating (i.e., the derivative of the right hand side with respect to N_t is always positive for $N_t > 0$; proof not required).

(a) Show that for any $S > 0$, the trivial equilibrium $\hat{N} = 0$ is asymptotically stable.

(b) Show that with increasing S , the system undergoes a fold bifurcation and find the critical value of S above which the population has no positive equilibrium. *Hint:* calculating the equilibrium as a function of S is rather cumbersome. It is easier to find the value of S necessary to have a certain equilibrium \hat{N} ; draw S against \hat{N} and reverse the axes to obtain a bifurcation diagram of the equilibria. The non-degeneracy conditions of the fold bifurcation don't have to be checked.