

Please include your *name* and *student number* on your answer sheets. In addition to standard writing equipment, you are allowed to bring in and consult **one** handwritten two-sided A4 sheet of personal notes during the exam.

Problem 1

As proven during the lectures, the Hilbert space $L^2([0, 2\pi])$ has a Hilbert basis $(e_k)_{k \in \mathbb{Z}}$ where $e_k(x) := \frac{1}{\sqrt{2\pi}} e^{ikx}$ for $x \in [0, 2\pi]$, $k \in \mathbb{Z}$. Suppose $f \in L^2([0, 2\pi])$ and denote its scalar product with e_k by $c_k := (f | e_k) \in \mathbb{C}$, $k \in \mathbb{Z}$.

(Grading: To get the maximum points from the questions in this Problem, it is enough to state the correct answer, you do not need to justify it. Maximum *2 points* from each of the items.)

- Express $\|f\|$ using only the numbers c_k , $k \in \mathbb{Z}$.
- By general Hilbert space theory, we can approximate f arbitrarily well by finite linear combinations which involve only c_k and e_k . Write down this approximation explicitly, including in which sense the “error” goes to zero.
- What does the previous result imply about the convergence of Fourier series of f ? Can you conclude that the series then converges at every point $x \in [0, 2\pi]$?

Problem 2

Let E be a Banach space, let $I = \text{id}_E$ denote the identity operator on E , and suppose $T \in \mathcal{L}(E)$.

- What is the Neumann series of T , under which standard condition on $\|T\|$ is it convergent, and which operator does it then converge to? (You do not need to prove your answer.) (*2 points*)
- Assume that the convergence condition in (a) is satisfied and show that then

$$\|(I - T)^{-1} - I - T\| \leq \frac{\|T\|^2}{1 - \|T\|}. \quad (4 \text{ points})$$

Problem 3

- What is the dual space E^* of a normed space E ? How is the canonical bilinear form $\langle x, y^* \rangle$ for $x \in E$, $y^* \in E^*$ defined? (*2 points*)
- Suppose $1 < p < \infty$. During the lectures it was proven that then $(\ell^p)^* \cong \ell^q$ for some real parameter q . Which q , and what does “ \cong ” mean here? (*1 point*)
- As above, suppose $E = \ell^p$ and $1 < p < \infty$. Prove that the rule

$$(Tx)_n = x_{n+1}, n \in \mathbb{N},$$

defines a bounded linear operator $T : E \rightarrow E$. Construct a bounded linear map $S : E^* \rightarrow E^*$ such that

$$\langle x, Sy^* \rangle = \langle Tx, y^* \rangle,$$

for all $x \in E$ and $y^* \in E^*$. (*3 points. Hint: You can try first solving the problem in the case $p = 2$ using the Fréchet–Riesz theorem.*)

Problem 4

- Write down the assumptions and statement of the closed graph theorem. (*2 points*)
- Suppose H is a Hilbert space, with an inner product $(\cdot | \cdot)$, and $T : H \rightarrow H$ is a linear map. Assume that $(Tx | y) = (x | Ty)$ for all $x, y \in H$. Show that T is a bounded operator. (*4 points. Hint: Closed graph theorem.*)