Department of Mathematics and Statistics
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**FUNCTIONAL ANALYSIS II EXAM , Feb. 8th, 2023**

No calculators or tables are allowed in the exam.
Solutions are accepted in English, Finnish or Swedish.

**Note! In Problems 2 and 4 you are expected to answer to only one of the cases a) or b); choose as you wish.**

**1.** Show that the subspace $X = \{f \in C^\infty(\mathbb{R}) \mid f(2) = 0\}$ of the space $C^\infty(\mathbb{R})$ is closed, in other words, if $(f_n)_{n=1}^\infty$ is a sequence of elements of $X$ converging to a function $g \in C^\infty(\mathbb{R})$, then $g \in X$.
The topology of $C^\infty(\mathbb{R})$ is defined by the seminorms

$$p_m(h) = \sup_{\substack{-m \le x \le m \\ k \le m}} \left| \frac{d^k h(x)}{dx^k} \right|,$$

where $m = 1, 2, \ldots$

**2.** Choose a) or b). Both items concern distributions on the domain $\Omega = \mathbb{R}$.

**a)** Prove that the Fourier-transform of the constant function $f(x) = 1 \ \forall x \in \mathbb{R}$, $f \in \mathcal{S}'(\mathbb{R})$, is a constant times the Dirac measure of 0.

**b)** Show that the function of the variable $x \in \mathbb{R}$

$$f_n(x) = \frac{\sqrt{n}}{2\sqrt{\pi}} e^{-nx^2/4}, \quad n \in \mathbb{N},$$

converges to the distribution $\delta_0$ in the weak topology of the space $\mathcal{D}'(\mathbb{R})$, when $n \to \infty$.

**3.** Let $\Omega = ]0, 5[$. Does the function $f(x) = (2-x)^{3/5}$, $x \in \Omega$, belong to the Sobolev space $W^{1,1}(\Omega)$, $W^{1,3}(\Omega)$ or $W^{3,1}(\Omega)$ ? Does the function $g(x) = (2-x)|2-x|$ belong to the Sobolev space $W^{2,2}(\Omega)$?

**4.** (Choose a) or b))
a) Let $\Pi \subset \mathbb{R}^2$ be a periodic domain in the plane as in Chapter 9 of the lectures. Present the definition of the Floquet-transform, its inverse and isomorphic mapping properties in the spaces $L^2(\Pi)$ and $H^1(\Pi) = W^{1,2}(\Pi)$. You do not need to prove anything.
b) The Sobolev embedding, i.e., the identity operator $I : W^{1,p}(\mathbb{R}) \to L^2(\mathbb{R})$ is continuous for all $1 < p < 2$. Show that the operator $I$ is not compact (it does not map all bounded subsets of the domain into precompact subsets of the target space, equivalently, there is a bounded sequence in the domain which does not have convergent subsequences in the target space) for any $p$, $1 < p < 2$.