FOURIER ANALYSIS I. **(Spring 2020)**

FINAL EXAM (Thursday 5.3 9-12 in room B121)

Choose freely only 4 questions out the following 6 !!

1. Let $f \in C^1_\sharp(-\pi,\pi)$ be a function whose all Fourier coefficients are real, i.e. $\widehat{f}(n) \in \mathbf{R}$ for all $n \in \mathbf{Z}$. Show that f is even, i.e. $f(-x) = f(x)$ for all $x \in (-\pi,\pi)$.

2. Let $f \in C^2_\#$ and $\int_{-\pi}^{\pi} f(x)dx = 0$. Prove Poincare type inequality.

$$\int_{-\pi}^{\pi} |f(x)|^2 dx \leq \int_{-\pi}^{\pi} |f''(x)|^2 dx.$$

For which functions do you have equality here?

3. Give an example of function $f \in L^1[-\pi,\pi)$ such that its Fourier series is not absolutely convergent but it converges at every point $x \in [-\pi,\pi)$. Do this by using the results of the lectures without actually computing the Fourier coefficients.

4. Find the Fourier coefficients of the 2π-periodic function f, where $f(x) = |x|$, for $[-\pi,\pi)$. At which points does the obtained Fourier series converge?

5. Solve – to find a formal solution formula is enough – by using Fourier series the following PDE. Here $x \in [0,2\pi)$, $t \geq 0$ and $x \to u(x,t)$ is assumed to be 2π-periodic, and the solution satisfies the initial value $u(x,0) = f(x)$, where $f \in L^2(-\pi,\pi)$ (of course you may assume that f is 2π-periodic).

$$\frac{d}{dt}u(x,t) = 4\Big(\frac{d}{dx}\Big)^2 u(x,t).$$

What happens to the solution as $t \to \infty$?

6. Recall the definition of equidistribution (mod 1) for given sequence $(x_n)_{n=1}^{\infty}$ of real numbers. What is Weyl's criterion for equidistribution? Use it to prove that $(n\alpha)_{n=1}^{\infty}$ is equidistributed if α is irrational.