Math. methods of phys. IIa Final exam 4.3.2020

On every paper, write your name, study number, course name and date

1. Consider the following first order quasilinear PDE

$$\partial_t u(x,t) + \partial_x u(x,t) + (t+x)u(x,t) = 0 \ .$$

   i. Construct the general solution and check your result.

   ii. Let us further impose the initial condition

$$u(x,0) = e^{-\frac{1}{2}x^2} \quad .$$

   What is the particular solution in this case?

2. A long cable has a rectangular cross-section, with width a and height b. The sides at $x = 0$ and $x = a$ are conducting, and earthed, meaning that the electrostatic potential $\phi = 0$ there. In the interior, ϕ obeys the Laplace equation

$$\nabla^2 \phi(x,y) = 0.$$

   i. Find the most general solution for ϕ consistent with the boundary conditions at $x = 0, a$.

   The potential on the sides $y = 0$ and $y = b$ is

$$\phi(x,0) = +V\frac{a}{2}\delta(x - a/2), \quad \phi(x,b) = -V\frac{a}{2}\delta(x - a/2)$$

   where $\delta(x)$ is the Dirac δ-function and V is a constant.

   ii. Find the potential $\phi(x,y)$ in the interior of the cable.

   Hint: use separation of variables. The following orthogonality integrals of trigonometric functions may be useful:

$$\int_0^L \mathrm{d}x \, \sin\left(\frac{\pi m}{L}x\right) \sin\left(\frac{\pi n}{L}x\right) = \frac{L}{2}\delta_{mn},$$

$$\int_0^L \mathrm{d}x \, \cos\left(\frac{\pi m}{L}x\right) \cos\left(\frac{\pi n}{L}x\right) = \frac{L}{2}\delta_{mn}, \quad m,n \geq 1$$

$$\int_0^L \mathrm{d}x \, \sin\left(\frac{\pi m}{L}x\right) \cos\left(\frac{\pi n}{L}x\right) = 0 \ .$$

3. Bessel's equation is

$$z^2\frac{\mathrm{d}^2 f}{\mathrm{d}z^2} + z\frac{\mathrm{d}f}{\mathrm{d}z} + (z^2 - \nu^2)f = 0 \,,$$

where $\nu \in \mathbb{R}$ and $\nu \geq 0$.

   i. What are the special points of the equation and what are their types?
   ii. Using the Frobenius/series method around the point $z = 0$, find the roots of the indicial equation and the recursion relation.
   iii. Show that the recursion relation is solved by the coefficients in the expansion

$$J_{\pm\nu}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{\Gamma(\pm\nu + k + 1)} \left(\frac{z}{2}\right)^{2k\pm\nu} .$$

4. The generating functions

$$g_m(t, x) = \frac{(2m-1)!!}{(1 - 2xt + t^2)^{m+1/2}}$$

can be used to generate associated Legendre functions

$$\frac{(2m-1)!!}{(1 - 2xt + t^2)^{m+1/2}} = \sum_{n=m}^{\infty} \Pi_n^m(x) t^{n-m},$$

where

$$\Pi_n^m(x) = \frac{1}{2^n n!} \frac{d^{n+m}}{dx^{n+m}}(x^2 - 1)^n$$

and the associated Legendre functions are $P_n^m(x) = (1 - x^2)^{m/2} \Pi_n^m(x)$.

Use the generating function (or otherwise) prove the recursion relations

   i. $(n - m + 1)P_{n+1}^m(x) - (2n + 1)xP_n^m(x) + (n + m)P_{n-1}^m(x) = 0$
   ii. $P_{n+1}^{m+1}(x) - P_{n-1}^{m+1}(x) = (2n + 1)(1 - x^2)^{1/2} P_n^m(x)$