COSMOLOGY II (PAP326) Exam 20.12.2022 Fall 2022

You may answer in Finnish, Swedish, or English. Return your solutions into two separate piles: one for problems 1 and 3; the other for problems 2, 4, and 5. Remember to put your name and student number on every separate sheet of paper. It is not enough to do calculations; an explanation about what is being done is needed also. Using a calculator is allowed. Exam time is 4 hours.

1. Demonstrate that the two conditions,

$$\varepsilon(\varphi) \equiv \frac{M_{\text{Pl}}^2}{2}\left(\frac{V'}{V}\right)^2 \ll 1 \qquad \text{and} \qquad |\eta(\varphi)| \equiv \left|M_{\text{Pl}}^2 \frac{V''}{V}\right| \ll 1,$$

are necessary conditions for the slow-roll approximation to be valid. Why are these conditions not sufficient?

2. The matter density 2-point correlation function is defined $\xi(\mathbf{r}) \equiv \langle \delta(\mathbf{x})\delta(\mathbf{x}+\mathbf{r})\rangle$. The galaxy 2-point correlation function $\xi_g(\mathbf{r})$ is defined as the excess probability of finding a galaxy at separation $\mathbf{r}$ from another galaxy: $dP \equiv \bar{n}\,[1+\xi_g(\mathbf{r})]\,dV$, where $\bar{n}$ is the mean galaxy number density, dV is a volume element that is a separation $\mathbf{r}$ away from a chosen reference galaxy, and dP is the probability that there is a galaxy within dV. Assume that the galaxy number density $n(\mathbf{x})$ is a biased tracer of matter density: $\delta_g \equiv \delta n/\bar{n} = b\delta \equiv b\delta\rho_m/\bar{\rho}_m$, where b is a constant. Show that ξ_g and ξ are related and find the relation between them.

3. Matter density perturbations $\delta \equiv \delta\rho_m/\bar{\rho}_m$ evolve according to the Jeans equation

$$\ddot{\delta}_{\mathbf{k}} + 2\frac{\dot{a}}{a}\dot{\delta}_{\mathbf{k}} + \frac{k^2}{a^2}\frac{\delta p_{\mathbf{k}}}{\bar{\rho}_m} - 4\pi G\bar{\rho}_m\delta_{\mathbf{k}} = 0,$$

when perturbations in other density components can be ignored. Find the solution for the Jeans equation for pressureless matter perturbations when there is no other energy component, but the universe has the open geometry ($K < 0$) and is totally curvature dominated, i.e., we assume $\Omega_0 = \Omega_m \ll 1$, so that we can use the curvature-dominated ($\Omega_0 = 0$) background solution, considering only scales $\ll$ curvature radius.

4. The CMB temperature anisotropy can be expanded in terms of spherical harmonics as

$$\frac{\delta T}{T_0}(\theta,\phi) = \sum a_{\ell m} Y_{\ell m}(\theta,\phi)\,.$$

The variance of the multipole coefficients $a_{\ell m}$ is called the angular power spectrum C_ℓ.

(a) Show that

$$\left\langle \left(\frac{\delta T(\theta,\phi)}{T_0}\right)^2 \right\rangle = \sum_\ell \frac{2\ell+1}{4\pi} C_\ell\,.$$

Explain the meaning of the angle brackets $\langle\cdot\rangle$ in this context.

(b) Show that

$$\frac{1}{4\pi}\int \left(\frac{\delta T(\theta,\phi)}{T_0}\right)^2 d\Omega = \sum_\ell \frac{2\ell+1}{4\pi}\widehat{C}_\ell\,.$$

Explain the meaning of the "hat" notation $\widehat{\cdot}$ in this context.

5. Write an essay (max 500 words) on (cosmological) inflation (no calculations). Grading of the essay will consider how well all important aspects were covered.

Table 1: The particles in the standard model of particle physics
Particle Data Group, 2022

Quarks	t	$172.69 \pm 0.30\,\text{GeV}$	$\bar{t}$	spin=$\frac{1}{2}$	$g = 2 \cdot 3 = 6$	
	b	4.16–4.21 GeV	$\bar{b}$	3 colors		
	c	$1.27 \pm 0.02\,\text{GeV}$	$\bar{c}$			
	s	90–102 MeV	$\bar{s}$			
	d	4.50–5.15 MeV	$\bar{d}$			
	u	1.90–2.65 MeV	$\bar{u}$			72
Gluons	8 massless bosons			spin=1	$g = 2$	16
Leptons	τ^-	$1776.86 \pm 0.12\,\text{MeV}$	τ^+	spin=$\frac{1}{2}$	$g = 2$	
	μ^-	105.658 MeV	μ^+			
	e^-	510.999 keV	e^+			12
	ν_τ	$< 1.1\,\text{eV}$	$\bar{\nu}_\tau$	spin=$\frac{1}{2}$	$g = 1$	
	ν_μ	$< 1.1\,\text{eV}$	$\bar{\nu}_\mu$			
	ν_e	$< 1.1\,\text{eV}$	$\bar{\nu}_e$			6
Electroweak gauge bosons	W^+	$80.377 \pm 0.012\,\text{GeV}$	W^-	spin=1	$g = 3$	
	Z^0	$91.1876 \pm 0.0021\,\text{GeV}$				
	γ	$0 \quad (< 1 \times 10^{-18}\,\text{eV})$			$g = 2$	11
Higgs boson	H^0	$125.25 \pm 0.17\,\text{GeV}$		spin=0	$g = 1$	1

$$g_f = 72 + 12 + 6 = 90$$
$$g_b = 16 + 11 + 1 = 28$$
$$g_* = \tfrac{7}{8} g_f + g_b = 106.75$$

Table 2: History of $g_*(T) \equiv 30\rho/(\pi^2 T^4)$

$T \sim 200$ GeV	all present	106.75	
$T \sim 100$ GeV	EW transition	(no effect)	
$T < 170$ GeV	top annihilation	96.25	
$T < 80$ GeV	$W^\pm$, Z^0, H^0	86.25	
$T < 4$ GeV	bottom	75.75	
$T < 1$ GeV	charm, τ^-	61.75	
$T \sim 150$ MeV	QCD transition	17.25	(u,d,s,g$\to \pi^{\pm,0}$, $47.5 \to 3$)
$T < 100$ MeV	$\pi^\pm$, π^0, μ^-	10.75	$e^\pm$, ν, $\bar{\nu}$, γ left
$T < 500$ keV	e^- annihilation	(7.25)	$2 + 5.33(4/11)^{4/3} = 3.38$

Taulukko 1: Standardimallin hiukkaset
Particle Data Group, 2022

Kvarkit	t	$172.69 \pm 0.30\,\text{GeV}$	$\bar{t}$	spin=$\frac{1}{2}$	$g = 2 \cdot 3 = 6$	
	b	4.16–4.21 GeV	$\bar{b}$	3 väriä		
	c	$1.27 \pm 0.02\,\text{GeV}$	$\bar{c}$			
	s	90–102 MeV	$\bar{s}$			
	d	4.50–5.15 MeV	$\bar{d}$			
	u	1.90–2.65 MeV	$\bar{u}$			
						72
Gluonit	8 massatonta bosonia			spin=1	$g = 2$	16
Leptonit	τ^-	$1776.86 \pm 0.12\,\text{MeV}$	τ^+	spin=$\frac{1}{2}$	$g = 2$	
	μ^-	105.658 MeV	μ^+			
	e^-	510.999 keV	e^+			
						12
	ν_τ	$< 1.1\,\text{eV}$	$\bar{\nu}_\tau$	spin=$\frac{1}{2}$	$g = 1$	
	ν_μ	$< 1.1\,\text{eV}$	$\bar{\nu}_\mu$			
	ν_e	$< 1.1\,\text{eV}$	$\bar{\nu}_e$			
						6
Sähköheikot mittabosonit	W^+	$80.377 \pm 0.012\,\text{GeV}$	W^-	spin=1	$g = 3$	
	Z^0	$91.1876 \pm 0.0021\,\text{GeV}$				
	γ	$0 \quad (< 1 \times 10^{-18}\,\text{eV})$			$g = 2$	
						11
Higgsin bosoni	H^0	$125.25 \pm 0.17\,\text{GeV}$		spin=0	$g = 1$	1

$$g_f = 72 + 12 + 6 = 90$$
$$g_b = 16 + 11 + 1 = 28$$
$$g_* = \tfrac{7}{8} g_f + g_b = 106.75$$

Taulukko 2: $g_*(T)$:n historia

$T \sim 200$ GeV	kaikki	106.75	
$T \sim 100$ GeV	EW-transitio	(ei vaikutusta)	
$T < 170$ GeV	top-annihilaatio	96.25	
$T < 80$ GeV	$W^\pm$, Z^0, H^0	86.25	
$T < 4$ GeV	b-kvarkki	75.75	
$T < 1$ GeV	lumo, τ^-	61.75	
$T \sim 150$ MeV	QCD-transitio	17.25	(u,d,s,g$\to \pi^{\pm,0}$, $47.5 \to 3$)
$T < 100$ MeV	$\pi^\pm$, π^0, μ^-	10.75	$e^\pm$, ν, $\bar{\nu}$, γ jäljellä
$T < 500$ keV	e^--annihilaatio	(7.25)	$2 + 5.33(4/11)^{4/3} = 3.38$

$1\,\mathrm{eV} = 11600\,\mathrm{K} = 1.60\times10^{-19}\,\mathrm{J} = 5.07\times10^{6}\,\mathrm{m}^{-1} = 1.52\times10^{15}\,\mathrm{s}^{-1} = 1.78\times10^{-36}\,\mathrm{kg}$

$c = 1 = 2.998\times10^8\,\mathrm{m/s}$

$\hbar = 1 = 197\,\mathrm{MeVfm}$

$1\,\mathrm{pc} = 3.09\times10^{16}\,\mathrm{m} = 3.26\,\mathrm{a}$

$1\,\mathrm{a} = 3.156\times10^7\,\mathrm{s}$

$h \equiv H_0/(100\,\mathrm{km/s/Mpc})$

$(100\,\mathrm{km/s/Mpc})^{-1} = 9.78\times10^9\,\mathrm{a} = 2998\,\mathrm{Mpc}$

$T_0 = 2.7255\,\mathrm{K} = 2.349\times10^{-4}\,\mathrm{eV}$

$T_{\nu0} = (4/11)^{1/3}T_0$

$z_{\mathrm{dec}} = 1090$

$m_e = 0.511\,\mathrm{MeV}$

$m_N = 938\,\mathrm{MeV}$

$m_p + m_e - m_H = 13.6\,\mathrm{eV}$

$\mathrm{M}_\odot = 1.99\times10^{30}\,\mathrm{kg}$

$\zeta(3) = 1.20206$

$m_{\mathrm{Pl}} \equiv G^{-1/2} = 1.22\times10^{22}\,\mathrm{MeV}$

$M_{\mathrm{Pl}} \equiv (8\pi G)^{-1/2} = 2.435\times10^{21}\,\mathrm{MeV}$

$g_n = g_p = g_e = 2, \qquad g_H = 4$

$Q = m_n - m_p = 1.293\,\mathrm{MeV}$

$g_*(T \ll m_e) = 3.384$

$g_{*S}(T \ll m_e) = 3.938$

$g_*(1\mathrm{MeV}) = g_{*S}(1\mathrm{MeV}) = 10.75$

$\tau_n = t_{1/2}/\ln 2 = 878\,\mathrm{s}$

$n + \nu_e \leftrightarrow p + e^-$

$\mu_e \ll T \qquad$ (kun/when $T > 30\,\mathrm{keV}$)

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-Kr^2} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right)$$

$$f(\vec p) = \frac{1}{e^{(E-\mu)/T} \pm 1}$$

$$n_i = \left\{ \begin{matrix} 1 \\ 3/4 \end{matrix} \right\} \frac{\zeta(3)}{\pi^2} g_i T^3, \quad (T \gg m_i)$$

$$n_i = g_i\left(\frac{m_i T}{2\pi}\right)^{3/2} e^{-\frac{m_i-\mu_i}{T}}, \quad (T \ll m_i)$$

$$\left(\frac{\dot a}{a}\right)^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\rho$$

$$\frac{\ddot a}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

$$\rho = \frac{\pi^2}{30} g_* T^4$$

$$s = \frac{2\pi^2}{45} g_{*S} T^3$$

$$g_*^{1/2} t T^2 = 0.301 m_{\mathrm{Pl}}$$

$$\rho = \frac{1}{2}\dot\varphi^2 + \frac{1}{2}\nabla\varphi^2 + V(\varphi)$$

$$\ddot\varphi - \frac{1}{a^2}\nabla^2\varphi + 3H\dot\varphi + V'(\varphi) = 0$$

$$n - \bar n = \frac{gT^3}{6\pi^2}\left[\pi^2\left(\frac{\mu}{T}\right) + \left(\frac{\mu}{T}\right)^3\right], \quad (T \gg m)$$

$$n - \bar n = 2g\left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T}\sinh\frac{\mu}{T}, \quad (T \ll m)$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec u) = 0$$

$$\frac{\partial\vec u}{\partial t} + (\vec u\cdot\nabla)\vec u + \frac{1}{\rho}\nabla p + \nabla\tilde\Phi = 0$$

$$\nabla^2\tilde\Phi = 4\pi G\rho$$

$$\mathcal{R} = -H\frac{\delta\varphi}{\dot\varphi}$$

$$\mathcal{P}_g(k) \equiv \left(\frac{L}{2\pi}\right)^3 4\pi k^3 \langle|g_{\mathbf k}|^2\rangle$$

$$\left(\frac{2\pi}{L}\right)^3 \sum_{\mathbf k} \to \int d^3k$$

$$\mathcal{P}_\varphi(k) = \left(\frac{H}{2\pi}\right)^2_{|aH=a_0k}$$

$$\mathcal{P}_{\mathcal R}(k) = \left(\frac{H}{\dot\varphi}\right)^2\left(\frac{H}{2\pi}\right)^2_{|aH=a_0k}$$

$$n_s(k) - 1 \equiv \frac{d\ln\mathcal{P}_{\mathcal R}}{d\ln k}$$

$$\left(\frac{\delta T}{T}\right)_{\mathrm{obs}} = \frac{1}{4}\delta_\gamma^N - \mathbf{v}^N\cdot\hat{\mathbf n} + \Phi(t_{\mathrm{dec}}, \mathbf{x}_{\mathrm{ls}}) + 2\int\dot\Phi dt$$

$$e^{i\mathbf k\cdot\mathbf x} = 4\pi\sum_{lm} i^l j_l(kx) Y_{lm}(\hat{\mathbf x}) Y^*_{lm}(\hat{\mathbf k})$$

$$\Phi_{\mathbf k} \equiv -\frac{3}{2}\left(\frac{aH}{a_0k}\right)^2\delta_{\mathbf k}$$

$$\Phi_{\mathbf k} = -\frac{3+3w}{5+3w}\mathcal{R}_{\mathbf k}, \quad (w \equiv p/\rho)$$

$$\int d\Omega\, Y^*_{lm}(\theta,\phi) Y_{l'm'}(\theta,\phi) = \delta_{ll'}\delta_{mm'}$$

$$\sum_m |Y_{lm}(\theta,\phi)|^2 = \frac{2l+1}{4\pi}$$

$$\int_0^\infty \frac{dz}{z} j_l(z)^2 = \frac{1}{2l(l+1)}$$

$$\int_{-\infty}^{\infty} e^{-ax^2}dx = \sqrt{\frac{\pi}{a}}$$